# $L^p$ convergence and large deviations for a supercritical multi-type branching process in a random environment

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# Summary

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   - Model
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# Summary

# Background

- Smith and Wilkinson (1969) : i.i.d. environment, extinction.
- Athreya and Karlin (1971) : stationary and ergodic environment, fundamental limit theorems.
- Critical and subcritical cases : survival probability and conditional limit theorems ($d \geq 1$).
- Supercritical case : moderate and large deviations : Single type case (d=1) : Buraczewski & Dyszewski (2020) for LD, Grama, Liu & Miqueu (2017) for MD.
  Multitype case ($d > 1$) : Grama, Liu & Pin (2020) for MD.

Here we focus on the supercritical case with $d > 1$, and study precise large deviation and $L^p$ convergence

# Summary

# Multi-type branching process

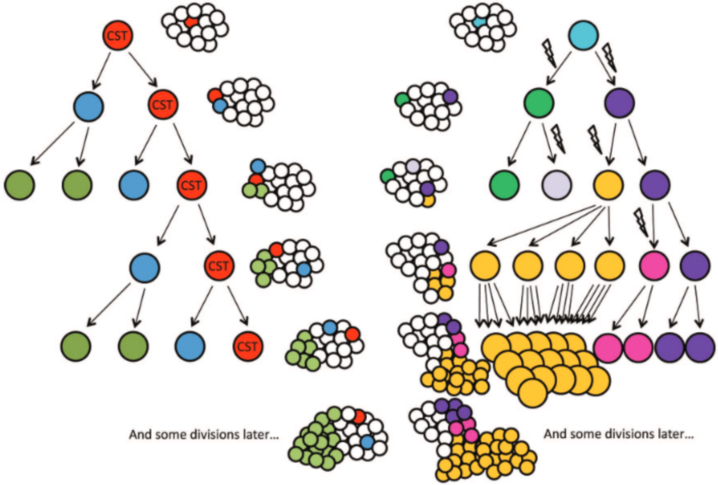


FIGURE – Division of a cancer cell

# Multi-type Galton-Watson process

A $d$-type branching process $Z_n = (Z_n(1), \cdots, Z_n(d))$ starting from $v \in \mathbb{N}^d \setminus \{0\}$ is defined as follows : $Z_0 = v$,

$$Z_{n+1} = \sum_{r=1}^{d} \sum_{l=1}^{Z_n(r)} N_{l,n}^r \quad n \geq 0.$$

- $Z_n(j) =$ number of particles of type $j$ in generation $n$ ;
- $N_{l,n}^r(j) =$ # children of type $j$ of $l$-th particle of type $r$, of gen. $n$.
- Galton-Watson process : all $N_{l,n}^r$ are independent, and have p.g.f. indep. of $n$ and $l$ : for $s = (s_1, \cdots, s_d) \in [0,1]^d$,

$$f^r(s) = \mathbb{E}\Big(\prod_{j=1}^{d} s_j^{N_{l,n}^r(j)}\Big) = \sum_{k_1,\cdots,k_d=0}^{\infty} p_{k_1,\cdots,k_d}^r s_1^{k_1} \cdots s_d^{k_d},$$

i.e. $\mathbb{P}(N_{l,n}^r = k) = p_k^r, \quad \forall k = (k_1, \cdots, k_d), n \geq 0, l \geq 1.$

# Multitype branching process in a random environment

The offspring distributions of gen. $n$ depend on the random environment $\xi_n$ at time $n$. We suppose that the random environment sequence $\xi = (\xi_0, \xi_1, \cdots)$ is i.i.d. Denote

$$\mathbb{P}_\xi = \mathbb{P}(\cdot|\xi) \text{ (quenched law)}, \quad \mathbb{E}_\xi = \mathbb{E}[\cdot|\xi]$$

Conditioned on $\xi$,

- the r.v.'s $N^r_{l,n}$ are independent for $l \geq 1$, $n \geq 0$, $1 \leq r \leq d$;
- each $N^r_{l,n}$ has prob. generating function depending on $\xi_n$ :

$$f^r_{\xi_n}(s) = \mathbb{E}_\xi\left(\prod_{j=1}^d s_j^{N^r_{l,n}(j)}\right) = \sum_{k_1,\cdots,k_d=0}^{\infty} p^r_{k_1,\cdots,k_d}(\xi_n) s_1^{k_1} \cdots s_d^{k_d}.$$

i.e. $\mathbb{P}_\xi(N^r_{l,n} = k) = p^r_k(\xi_n), \ \forall k = (k_1, \cdots, k_d), n \geq 0, l \geq 1.$

$Z_n$ reduces to the Galton-Watson process if $\xi_n = c$ (const.) $\forall n$.

# Lyapunov exponent for products of mean matrices

For $n, k \geq 0$, consider $M_{k,n} := M_k \cdots M_n$, where $M_n$ is the mean matrix of the offspring distributions at gen. $n$ :

$$M_n(i,j) = \mathbb{E}_\xi \left[ Z_{n+1}(j) \middle| Z_n = e_i \right] = \frac{\partial f^i_{\xi_n}}{\partial s_j}(1)$$

$$= \mathbb{E}_\xi \# \{\text{children of type j of a type i particle of gen. n}\}$$

Assume $\mathbb{E} \log^+ \|M_0\| < +\infty$. The Lyapunov exponent of $(M_n)$ is defined by

$$\gamma = \lim_{n \to +\infty} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\| = \inf_{n \geq 1} \frac{1}{n} \mathbb{E} \log \|M_{0,n-1}\|.$$

$$\gamma = \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}\log\|M_{0,n-1}\| = \inf_{n\geq 1}\frac{1}{n}\mathbb{E}\log\|M_{0,n-1}\|.$$

## Classification

We say that the multi-type branching process $(Z_n)$ in the random environment $\xi$ is :

- sub-critical if $\gamma < 0$ $\quad (\|Z_n\| \underset{n\to+\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.})$
- critical if $\gamma = 0$ $\quad (\|Z_n\| \underset{n\to+\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.})$
- supercritical if $\gamma > 0$ $\quad (\mathbb{P}(\|Z_n\| \underset{n\to+\infty}{\longrightarrow} +\infty) > 0.)$

Here we only consider the supercritical regime, i.e. $\gamma > 0$.

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# Objective

We write $Z_n^i$ for $Z_n$ when $Z_0 = e_i$, that is, when the process starts with one initial particle of type $i$.
We focus on the supercritical regime ($\gamma > 0$), and search for :

- asymptotic properties of

$$Z_n^i(j) = \#\text{ particles of type j of gen. n, when } Z_0 = e_i$$

  by studying the $L^p$ convergence of $\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i,j)}$ ;
- asymptotic properties of

$$\|Z_n\| = Z_n(1) + \cdots + Z_n(d) = \text{ population size of gen. n}$$

  by establishing a precise large deviation result of type Bahadur-Rao : for $q > \gamma$,

$$\mathbb{P}(\log \|Z_n\| > qn) \sim ?$$

# Summary

# Furstenberg- Kesten condition, law of large numbers

**Condition FK** (Furstenberg- Kesten)

There exists a constant $C > 1$ such that

$$1 \leq \frac{\max_{1\leq i,j\leq d} M_0(i,j)}{\min_{1\leq i,j\leq d} M_0(i,j)} \leq C \quad \mathbb{P}\text{-a.s.}$$

Recall $\quad \gamma = \lim_{n\to+\infty} \frac{1}{n}\mathbb{E}\log\|M_{0,n-1}\| \quad$ (Lyapunov exponent)

**LLN for the components $M_{0,n-1}(i,j)$ (Furstenberg-Kesten 1960)**

Assume condition **FK** and $\mathbb{E}\log^+\|M_0\| < +\infty$. Then

$$\forall i,j \lim_{n\to+\infty} \frac{1}{n}\log M_{0,n-1}(i,j) = \gamma \quad \mathbb{P}\text{-a.s.}$$

## Kesten-Stigum type theorem (Grama-Liu-Pin, AAP 2023)

Assume condition **FK**, $\gamma > 0$, and $\xi = (\xi_n)_{n\geq 0}$ i.i.d. Then there exist random variables $W^i \in [0,\infty)$ such that for all $1 \leq j \leq d$,

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \xrightarrow[n\to+\infty]{\mathbb{P}} W^i$$

Moreover, $\max_{1\leq i\leq d} \mathbb{P}(W^i = 0) < 1$ ($W^i$ non-degenerate) iff

$$\max_{1\leq i,j\leq d} \mathbb{E}\left(\frac{Z_1^i(j)}{M_0(i,j)} \log^+ \frac{Z_1^i(j)}{M_0(i,j)}\right) < +\infty. \quad (*)$$

When $(*)$ holds, then for all $1 \leq i \leq d$, a.s. $\mathbb{E}_\xi W^i = 1$ and

$$\{W^i = 0\} = \{\|Z_n^i\| \underset{n\to+\infty}{\to} 0\}, \quad \{W^i > 0\} = \{\|Z_n^i\| \underset{n\to+\infty}{\to} +\infty\}.$$

A similar version is found for stationary and ergodic environment.

# Remarks

1. (G-W case) When $\xi_0 = \xi_1 = \cdots = c$ (*const*.), it is equivalent to the theorem of Kesten-Stigum (1966) , as

$$M^n(i,j) \sim \rho^n u(i)v(j),$$

where $M$ is the mean matrix with spectral radius $\rho > 1$, $u, v$ are the right and left eigenvectors ($Mu^T = \rho u^T, vM = \rho v$) with $u, v > 0, \|u\| = \langle u, v\rangle = 1$ (Perron-Frobenius Th).

2. (Single type case) When $d = 1$, due to Athreya-Karlin (1971, sufficiency) and Tanny (1988, necessity)
3. It is a bridge linking MBPRE and products of random matrices.
4. It gives the exponential increasing of $Z^i(j)$ since it implies

$$\lim_{n\to+\infty} \frac{1}{n} \log Z_n^i(j) = \gamma > 0 \quad \mathbb{P}\text{-a.s.}$$

# Summary

# $L^p$-convergence of the normalized population : NSC

$$I := \left\{ s \in \mathbb{R} : \max_{1 \le i,j \le d} \mathbb{E} M_0(i,j)^s < +\infty \right\},$$

$$\kappa(s) := \lim_{n \to +\infty} \left( \mathbb{E} \| M_{0,n-1} \|^s \right)^{1/n} \quad \text{exists, is finite for } s \in I.$$

## Theorem (Grama-Liu-Pin 2022)

Assume (FK). Let $p > 1$ be such that $1 - p \in I$. Then

$$\bar{Z}_n^i(j) := \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \xrightarrow{L^p} W^i \quad \text{for all } 1 \le i, j \le d$$

if and only if

$$\max_{1 \le i,j \le d} \mathbb{E} \left( \frac{Z_1^i(j)}{M_0(i,j)} \right)^p < +\infty \quad \text{and} \quad \kappa(1-p) < 1.$$

# $L^p$-convergence of $(\bar{Z}_n^i(j))$ : historic notes

- For $d \geq 1$, Cohn (Ann. Prob. 1989) claimed that if for some constants $c_1, C_1, C_2 > 0$,

$$c_1 \leq M_0(i,j) \leq C_1, \quad \mathbb{E}_\xi (Z_1^i(j))^2 \leq C_2$$

and $\mathbb{E}|\log \sum_{i=1}^d (1 - \mathbb{P}(\|Z_1^i\| = 0))| < \infty$, then

$$\frac{Z_n^i(j)}{\mathbb{E}_\xi Z_n^i(j)} \text{ converges in } L^2.$$

By our theorem, the essential quantitative condition $\kappa(-1) < 1$ is missing in his claim.

- For $d = 1$, Afanasyev (2001) obtained a sufficient condition (which is not necessary) ; Guivarc'h and Liu (2001) got the NSC : with $M_n =$ mean of the offspring distribution of gen. n,

$$\frac{Z_n}{M_{0,n-1}} \text{ conv. in } L^p \Leftrightarrow \mathbb{E}\left(\frac{Z_1}{M_0}\right)^p < +\infty \text{ and } \mathbb{E}M_0^{1-p} < 1.$$

# Exponential $L^p$-convergence

**Theorem (Grama-Liu-Pin 2022)**

There exist $\delta \in (0, 1)$ and $C \in (0, \infty)$ such that $\forall n \geq 1$,

$$\mathbb{E}\left|\bar{Z}_n^i(j) - W^i\right|^p \leq C\delta^n$$

- We have an expression of $\delta$, but the optimal value of $\delta$ is not known.
- For $d = 1$, Huang and Liu (2014) found the optimal value of $\delta$.

# Summary

# The difficulty for the proof of $L^p$ convergence of $\bar{Z}_n^i(j)$

- $(\bar{Z}_n^i)$ is not a martingale.
- To overcome the difficulty, we study the $L^p$ convergence of the fundamental martingale which has similar properties : Assume $M_0 > 0$ a.s. For $0 \leq n \leq k$, let $\rho_{n,k}$ = spectral radius of $M_{n,k}$, $U_{n,k} > 0$ and $V_{n,k} > 0$ be the right and left eigenvectors :

$$M_{n,k}\ U_{n,k}^T = \rho_{n,k}\ U_{n,k}^T, \quad V_{n,k} M_{n,k} = \rho_{n,k}\ V_{n,k},$$

such that $\|U_{n,k}\| = \langle U_{n,k}, V_{n,k} \rangle = 1$. By Hennion (1997),

$$U_{n,\infty} = \lim_{k \to +\infty} U_{n,k} > 0, \quad M_n U_{n+1,\infty}^T = \lambda_n U_{n,\infty}^T$$

where $\|U_{n,\infty}\| = 1$, $(U_{n,\infty})$ and $(\lambda_n)$ are stationary and ergodic. By Grama-Liu-Pin (2023), with $\lambda_{0,n-1} = \pi_{i=0}^{n-1} \lambda_i$,

$$W_n^i = \frac{\langle Z_n^i, U_{n,\infty} \rangle}{\lambda_{0,n-1} U_{0,\infty}(i)} \text{ is a martingale, } \quad W_n^i \xrightarrow{a.s.} W^i.$$

# Proof of the $L^p$ convergence of $W_n^i$

1. We start with $1 < p \leq 2$. We show that
$$\mathbb{E}_\xi |W_{n+1}^i - W_n^i|^p \leq C\sigma_n(p)(\lambda_{0,n-1}U_{0,\infty}(i))^{1-p},$$
with $C > 0$ a constant, $\sigma_n(p) = \max_{1\leq j\leq d} \mathbb{E}_\xi \left| W_{n,1}^j - 1 \right|^p$, and $(W_{n,k}^j)_{k\geq 0}$ the martingale associated to a branching process starting with one particle of type $j$ in the shifted environment $T^n\xi = (\xi_n, \xi_{n+1}, \cdots)$.
2. We take the expectation conditioned on $T^n\xi$ :
$$\mathbb{E}_{T^n\xi} |W_{n+1}^i - W_n^i|^p \leq C\sigma_n(p)\mathbb{E}_{T^n\xi}(\lambda_{0,n-1}U_{0,\infty}(i))^{1-p}.$$
3. We show that for all $s \in I$, there is a constant $C_s$ such that
$$\begin{aligned}\mathbb{E}_{T^n\xi}(\lambda_{0,n-1}U_{0,\infty}(i))^s &= \mathbb{E}_{T^n\xi}\langle M_{0,n-1}U_{n,\infty}, e_i\rangle^s \\ &\leq \sup_{\|x\|=\|y\|=1} \mathbb{E}\langle M_{0,n-1}x, y\rangle^s \\ &\leq C_s\kappa(s)^n.\end{aligned}$$

# Proof of the $L^p$ convergence of $W_n^i$

4. We get

$$\mathbb{E}|W_{n+1}^i - W_n^i|^p \leq C\mathbb{E}\sigma_0(p)\kappa(1-p)^n.$$

5. For $p > 2$, we use an argument by induction.
6. To show that the condition is necessary, consider the transfer operator $P_s$ such that for $s \in I$ and all $\varphi \in \mathcal{C}(\mathcal{S})$,

$$P_s\varphi(x) := \mathbb{E}\left[\|xM_0\|^s \varphi(x \cdot M_0)\right], \quad x \in \mathcal{S},$$

where $x \cdot M_0 = xM_0/\|xM_0\|$ is the direction of the vector $xM_0$. We prove that $\kappa(s)$ is the spectral radius of $P_s$, and there exists a strictly positive eigenfunction $r_s \in \mathcal{C}(\mathcal{S})$ :

$$P_s r_s = \kappa(s) r_s.$$

Using the properties of the eigenfunction $r_{1-p}$ and the recursive relation satisfied by $W^i$ (due to the branching property), we obtain $\kappa(1-p) < 1$ (strict inequality).

# Conditions for products of positive random matrices

Denote by $\Gamma_\mu = [supp\, \mu]$ the smallest closed semigroup generated by $g$ in the support of $\mu$. Usual conditions :

1. (A1) Allowability : every $g \in \Gamma_\mu$ has the property that every row and every column of $g$ has a strictly positive entry.
2. (A2) Positivity : $\Gamma_\mu$ contains at least one strictly positive matrix.
3. (A3) Non-arithmeticity : the measure $\mu$ is non-arithmetic. The measure $\mu$ is said to be *arithmetic*, if $\exists t > 0$, $\beta \in [0, 2\pi)$ and a real function $\vartheta$ defined on $\mathcal{S} := \{v \in \mathbb{R}_+^d : \|v\| = 1\}$ such that for any $g \in \Gamma_\mu$ and any $x \in V(\Gamma_\mu)$,

$$\exp[it \log |gx| - i\beta + i\vartheta(g{\cdot}x) - i\vartheta(x)] = 1,$$

where $V(\Gamma_\mu) = \overline{\{v_g \in \mathcal{S} : g \in \Gamma_\mu, g \text{ is proximal}\}}$. (denote by $\lambda_g$ the dominant eigenvalue, $v_g$ the associated eigenvector with unit norm). This coincides with the usual arithmetic property when $d = 1$ : $\log |g_1| \in (\beta + 2\pi\mathbb{Z})/t$ a.s..

# Summary

# The functions $\kappa(s)$ and $\Lambda^*$

Set $\mathcal{S} = \{v \in \mathbb{R}_+^d : \|v\| = 1\}$,

$$I = \{s \in \mathbb{R} : \mathbb{E}(\|M_0\|^s) < \infty\}$$

For any $s \in I$ the limit

$$\kappa(s) = \lim_{n\to\infty} \left(\mathbb{E}\|M_{0,n-1}\|^s\right)^{\frac{1}{n}} \quad (= \mathbb{E}M_0^s = \mathbb{E}e^{s\log M_0} \text{ when } d = 1)$$

exists and is finite. Introduce the Fenchel-Legendre transform of the convex function $\Lambda(s) = \log\kappa(s), s \in I$ by

$$\Lambda^*(q) = \sup_{s\in I}\{sq - \Lambda(s)\}, \quad q \in \mathbb{R}.$$

We have $\Lambda^*(q) = sq - \Lambda(s)$ if $q = \Lambda'(s)$ for some $s \in I$.

# Bahadur-Rao type large deviations

## Theorem

Let $0 < s \in I^o$ (interior of I) and $q = \Lambda'(s)$. Assume (A1), (A2), (A3) and that one of the following conditions holds :

(H1) $s < 1$, $\mathbb{E}\left[\mathbb{E}\left[\|Z_1\|^{\frac{s}{s-\delta_s}} \mid \xi\right]^{s-\delta_s}\right] < \infty$ for some $\delta_s \in (0, s)$;

(H2) $s = 1$ and $\mathbb{E}\left[\|Z_1\|^{1+\delta_1}\right] < \infty$ for some $\delta_1 > 0$;

(H3) $s > 1$ and $\exists \delta_s > 0$ such that $\mathbb{E}\left[\|Z_1\|^{s+\delta_s}\right] < \infty$ and

$$\max_{1 \le i,j \le d} \mathbb{E}\left(\frac{\|M_0\|^{s+\delta_s}}{M_0(i,j)^{s+\delta_s}} Z_1^i(j)^{s+\delta_s}\right) < \infty.$$

Then there is a constant $\mathcal{C}_i(s) \in (0, \infty)$ such that

$$\mathbb{P}\left[\|Z_n^i\| \ge e^{nq}\right] \sim \frac{\mathcal{C}_i(s)}{\sqrt{n}} e^{-n\Lambda^*(q)} \quad \text{as} \quad n \to \infty.$$

$d = 1$ : due to Buraczewski-Dyszewski (2020).

# Summary

# Why true ?

1. Intuition : if $Z_0 = v$, $Z_n$ behaves like $\mathbb{E}_\xi Z_n = vM_{0,n-1}$,
2. More precisely, there is a bridge between the branching process $(Z_n^i)$ and the products of random matrices $M_{0,n-1}$, due to the Kesten-Stigum type theorem (Grama-Liu-Pin AAP 2023) :

$$\frac{Z_n^i(j)}{E_\xi Z_n^i(j)} = \frac{Z_n^i(j)}{M_{0,n-1}(i,j)} \xrightarrow{P} W_i \in (0,\infty) \text{ on } \{|Z_n| \to \infty\}$$

3. For large deviations and Gaussian approximation on products of random matrices, see Xiao-Grama-Liu (SPA 2020, JEMS 2022, AoP 2023).

# Summary

# Proof of large deviations

For the proof, we need a measure change argument of Cramer type and a $L^p$ convergence theorem for $Z_n$ under the changed measure.

# The operator $P_s$, eigenmeasure $\nu_s$ & eigenfunction $r_s$

For any $s \in I$, define the transfer operator $P_s$ : for any bounded measurable function $\varphi$ on $\mathcal{S}$,

$$P_s\varphi(x) := \mathbb{E}\Big[\|xM_0\|^s\varphi(x \cdot M_0)\Big], \quad x \in \mathcal{S}.$$

The operator $P_s$ has a unique probability eigenmeasure $\nu_s$ on $\mathcal{S}$ and a unique (up to a scaling constant) strictly positive and continuous eigenfunction $r_s$ on $\mathcal{S}$, corresponding to the eigenvalue $\kappa(s)$ :

$$P_s r_s = \kappa(s) r_s, \quad P_s \nu_s = \kappa(s)\nu_s.$$

# Measure change formula

Set

$$q_n^s(v, M_{0,n-1}) = \frac{e^{s \log \|vM_{0,n-1}\|}}{\kappa(s)^n} \frac{r_s(v \cdot M_{0,n-1})}{r_s(v)}.$$

Since $P_s r_s = \kappa(s) r_s$, $\int q_n^s(v, M_{n-1} \dots M_0) d\mathbb{P} = 1$, for any $n \geq 1$.
So $q_n^s(v, M_{0,n-1}) d\mathbb{P}$ is a probability measure on

$$\mathcal{F}_n = \sigma\{\xi_k, N_{k,j}^r : 1 \leq k < n, j \geq 1, 1 \leq r \leq d\}, \text{ with } \mathcal{F}_0 = \{\emptyset, \Omega\}.$$

By the Kolmogorov extension theorem, it can be extended to a probability measure, say $\mathbb{Q}_s^v$, on $(\Omega, \mathcal{F})$, so that for any bounded $\mathcal{F}_n$-measurable random variable $Y$, we have

$$\mathbb{E}_{\mathbb{Q}_s^v}(Y) = \mathbb{E}\Big[q_n^s(v, M_{0,n-1}) Y\Big]. \tag{1}$$

# $L^p$ convergence under $\mathbb{Q}_s^v$

For any $s \in I$, $v \in \mathcal{S}$ and $z \in \mathbb{C}$ with $s + \Re z \in I$, let

$$\kappa_s(z) := \lim_{n\to\infty} \left(\mathbb{E}_{\mathbb{Q}_s^v} \|M_{0,n-1}\|^z\right)^{\frac{1}{n}} = \frac{\kappa(s+z)}{\kappa(s)}.$$

**Theorem 2.**

Assume condition **(FK)**. Let $p > 1$ be such that $1 - p \in I$. If

$$\max_{1\le i,j\le d} \mathbb{E}_{\mathbb{Q}_s^{e_i}} \left(\frac{Z_1^i(j)}{M_0(i,j)}\right)^p < +\infty \quad \text{and} \quad \kappa_s(1-p) < 1, \qquad (2)$$

then $\dfrac{Z_n^i(j)}{(M_0 \cdots M_{n-1})(i,j)} \xrightarrow{n\to+\infty} W_s^i$ in $L^p(\mathbb{Q}_s^{e_i})$.

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**Thanks for your attention !** **quansheng.liu@univ-ubs.fr**